A CHARACTERIZATION OF N-DIMENSIONAL DAUBECHIES TYPE TENSOR PRODUCT WAVELET*

LI DENGFENG (李登峰)

(*Department of Mathematics, Henan University, Kaifeng* 475001, *China*)

and

(*National Mobile Communications Research Laboratory, Southeast University, Nanjing* 210096, *China*)

PENG SILONG (彭思龙)

(*Institute of Automation, Chinese Academy of Sciences, Beijing* 100080, *China*)

Abstract

In this paper, we consider the problem of the existence of general non-separable variate orthonormal compactly supported wavelet basis when the symbol function has a special form. We prove that the general non-separable variate orthonormal wavelet basis doesn't exist if the symbol function possesses a certain form. This helps us to explicate the difficulty of constructing the non-separable variate wavelet basis and to hint how to construct non-separable variate wavelet basis.

Key words. Daubechies type wavelet, symbol function, tensor product, orthonormality

1. Introduction

Since the introduction by Daubechies[1] of compactly supported orthogonal wavelet bases in R with arbitrary high smoothness, various new wavelet bases have been constructed and applied successfully in image processing, numerical computation, statistics, etc. Many of these applications, such as image compression, employ wavelet bases in R^2. Virtually all of these bases are separable, that is, the bivariate functions are simply tensor products of univariate basis functions. A separable varaite wavelet basis is easy to construct and simple to study, for it inherits the features of the corresponding wavelet basis in R, such as smoothness and support size.

Nevertheless, separable variate bases have a number of drawbacks. Because they are so special, they have very little design freedom. Further, separability imposes an unnecessary product structure, which is artificial for natural images. For example, the zero set of a

Received May 7, 1999. Revised January 31, 2000.

* This research is supported by the National Natural Science Foundation of China (No.69982002) and the Opening Foundation of National Mobile Communications Research Laboratory in Southeast University.

separable scaling function contains horizontal and vertical lines. This "preferred directions" effect can create unpleasant artifacts that become obvious at high image compression ratios. Nonseparable wavelet bases offer the hope of a more isotropic analysis[2,3]. Therefore many researchers want to construct nonseparable variate wavelet bases to adapt to practical necessity. In the last several years, only a few non-separable variate wavelet bases have been constructed[2,4–8]. Recently, the named second-named author gave a conclusion about non-separable variate orthonormal basis in R^2. His result is not only of theoretical interest, but also of interest in practical applications[3].

In this paper, we extend the result in [3] to the general case, that is, in R^n. In the result, we have found that the general non-separable variate wavelet basis does not exist when the corresponding symbol function has a certain form. This also shows why we can not construct non-separable variate wavelet bases easily.

The main result is in Section 2. Several auxiliary lemmas are given in Section 3. The proof of the result is show in Section 4.

2. Main Result

It is well known that the method to construct wavelets is usually via the founding of multiresolution analysis. Now we give the definition of multiresolution analysis.

Definition. A multiresolution analysis (abbr. MRA) consists of a sequence of closed subspaces V_j $(j \in Z)$ of $L^2(R^n)$ satisfying

- $V_j \subset V_{j+1}$ for all $j \in Z$;
- $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in Z$, $x \in R^n$;
- $\bigcap\limits_{j\in Z} V_j = \{0\}$; $\overline{\bigcup\limits_{j\in Z} V_j} = L^2(R^n)$;
- There exists a function $\phi \in V_0$, such that $\{\phi(\cdot - k)\}_{k\in Z^n}$ is an orthonormal basis for V_0, where $R^n = \{x \,|\, x = (x_1, \cdots, x_n),\ x_j \in R,\ j = 1, \cdots, n\}$, R is the set of all real numbers and $Z^n = Z \times \cdots \times Z$, Z is the set of all integers. The function ϕ whose existence is asserted in the above definition is called a scaling function of the given MRA.

Let $\{V_j\}_{j\in Z}$ be an MRA. Then there exists a periodic function $m_0(\xi_1, \cdots, \xi_n)$ such that

$$\widehat{\phi}(2\xi_1, \cdots, 2\xi_n) = m_0(\xi_1, \cdots, \xi_n)\widehat{\phi}(\xi_1, \cdots, \xi_n),$$

where $\widehat{\phi}$ denotes the Fourier transform of ϕ, that is,

$$\widehat{\phi}(\xi) = \int_{R^n} \phi(x) e^{-ix\cdot\xi}\, dx$$

and $\xi = (\xi_1, \cdots, \xi_n) \in R^n$. The function m_0 is called the symbol function (or low pass filter) associated with the scaling function ϕ.

The orthonormality of $\{\phi(\cdot - k)\}_{k\in Z^n}$ implies that m_0 satisfies

$$\sum_{\nu\in E^n} \left|m_0(\xi + \pi\nu)\right|^2 = 1, \tag{1}$$

where E^n denotes the set of all vertices of the unit cube $[0, 1]^n$ in R^n.

In real line R, I. Daubechies constructed compactly supported orthonormal wavelets which are called Daubechies type wavelets. The scaling function corresponding to Daubechies type wavelet is called Daubechies type scaling function. The corresponding symbol function is

$$m_0(\eta) = \Big(\frac{1 + e^{-i\eta}}{2}\Big)^N Q(e^{-i\eta}),$$

where $Q(x) = \sum_{k=0}^{N-1} q_k x^k$ is a polynomial, $q_k \in R$, $N \in Z^+$, Z^+ is the set of all natural numbers and $\exp(\eta) = e^\eta$. For details see [1].

Our main theorem is as follows.

Theorem. Suppose that $\phi(x_1, \cdots, x_n)$ is the scaling function corresponding to symbol function $m_0(\xi_1, \cdots, \xi_n)$ in a given MRA. Then $\phi(x_1, \cdots, x_n)$ is obtained by tensor product of Daubechies type scaling functions if and only if

$$m_0(\xi_1, \cdots, \xi_n) = \prod_{j=0}^{n} \Big(\frac{1+\exp(-i\xi_j)}{2}\Big)^{N_j} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_n=0}^{N_n-1} a_{k_1 \cdots k_n} \exp\Big(-i\sum_{l=1}^{n} k_l \xi_l\Big), \tag{2}$$

where $a_{k_1 \cdots k_n} \in R$ and for $j = 1, \cdots, n$, $N_j \in Z^+$

This theorem indicates that we should choose an m_0 to have a form different from (2) in order to construct a non-separable variate wavelet basis.

3. Lemmas

In order to prove Theorem, we now give several auxiliary lemmas. Denote $E_1^n = \{\nu \mid \nu = (\nu_1, \cdots, \nu_n) \in E^n, \sum_{j=1}^{n} \nu_j \text{ is even}\}$.

Lemma 1. $\forall\, k = (k_1, \cdots, k_n) \in Z^n$, $\xi = (\xi_1, \cdots, \xi_n) \in R^n$, we have

$$\cos(k \cdot \xi) = \sum_{\nu=(\nu_1,\cdots,\nu_n)\in E_1^n} d_\nu^k \prod_{j=1}^{n} \cos\Big(\nu_j \frac{\pi}{2} - k_j \xi_j\Big), \tag{3}$$

where d_ν^k's are constants.

Proof. Denote $\lambda_j = (0, \cdots, 0, k_j, 0, \cdots, 0)$, where k_j is the jth coordinate of λ_j. We use induction about the dimensionality n. By simple calculation, we know that (3) holds when $n = 1$ and $n = 2$. Now, assume that (3) holds when $n \le m$, we will show that the conclusion holds when $n = m + 1$ in the following:

$\forall\, k = (k_1, \cdots, k_{m+1})$, $\xi = (\xi_1, \cdots, \xi_{m+1})$, we have

$$\begin{aligned}
&\cos(k \cdot \xi) = \cos\big[(k - \lambda_{m+1}) \cdot \xi + \lambda_{m+1} \cdot \xi\big] \\
&= \cos\big[(k - \lambda_{m+1}) \cdot \xi\big] \cos(\lambda_{m+1} \cdot \xi) - \sin\big[(k - \lambda_{m+1}) \cdot \xi\big] \sin(\lambda_{m+1} \cdot \xi) \\
&= \cos\big[(k - \lambda_{m+1}) \cdot \xi\big] \cos(\lambda_{m+1} \cdot \xi) - \sin\big[(k - \lambda_{m+1} - \lambda_m) \cdot \xi + \lambda_m \cdot \xi\big] \sin(\lambda_{m+1} \cdot \xi) \\
&= \cos\big[(k - \lambda_{m+1}) \cdot \xi\big] \cos(\lambda_{m+1} \cdot \xi) - \sin\big[(k - \lambda_{m+1} - \lambda_m) \cdot \xi\big] \cos(\lambda_m \cdot \xi) \sin(\lambda_{m+1} \cdot \xi) \\
&\quad - \cos\big[(k - \lambda_{m+1} - \lambda_m) \cdot \xi\big] \sin(\lambda_m \cdot \xi) \sin(\lambda_{m+1} \cdot \xi) \\
&= \cos\big[(k - \lambda_{m+1}) \cdot \xi\big] \cos(\lambda_{m+1} \cdot \xi) + \frac{1}{2}\big[\cos(k - \lambda_m) \cdot \xi\big] \cos(\lambda_m \cdot \xi) \\
&\quad - \frac{1}{2} \cos\big[(k - 2\lambda_{m+1} - \lambda_m) \cdot \xi\big] \cos(\lambda_m \cdot \xi) - \cos\big[(k - \lambda_{m+1} - \lambda_m) \cdot \xi\big] \\
&\quad \cdot \sin(\lambda_m \cdot \xi) \sin(\lambda_{m+1} \cdot \xi).
\end{aligned}$$

Note that $\cos\big[(k - \lambda_{m+1}) \cdot \xi\big]$, $\cos\big[(k - \lambda_{m+1}) \cdot \xi\big]$, $\cos\big[(k - 2\lambda_{m+1} - \lambda_m) \cdot \xi\big]$ and $\cos\big[(k - \lambda_{m+1} - \lambda_m) \cdot \xi\big]$ can be expressed by the form (3), hence we deduce that $\cos(k \cdot \xi)$ can be also expressed by the form (3). The proof of Lemma 1 is finished.

Lemma 2. For a trigonometric polynomial

$$A(\xi_1, \cdots, \xi_n) = \sum_{k \in \Lambda} a_k \exp(-ik \cdot \xi), \qquad \xi = (\xi_1, \cdots, \xi_n),$$

where $a_k \in R$, $\Lambda = \{k = (k_1, \cdots, k_n) \mid k_j \in \Lambda_j\}$ and $\Lambda_j = \{0, \cdots, 2N_j - 1\}$ $(j = 1, \cdots, n)$, we have the following result:

$$\left|A(\xi_1, \cdots, \xi_n)\right|^2 = \sum_{k\in\Lambda} \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n} \prod_{j=1}^{n} b_\mu^k \cos\left(\mu_j \frac{\pi}{2} - k_j\xi_j\right). \tag{4}$$

Proof. Since

$$\begin{aligned}\left|A(\xi_1, \cdots, \xi_n)\right|^2 &= \Big(\sum_{k\in\Lambda} a_k \cos(k\cdot\xi)\Big)^2 + \Big(\sum_{k\in\Lambda} a_k \sin(k\cdot\xi)\Big)^2 \\ &= \sum_{k\in\Lambda} a_k \cos(k\cdot\xi) \sum_{l\in\Lambda} a_l \cos(l\cdot\xi) + \sum_{k\in\Lambda} a_k \sin(k\cdot\xi) \sum_{l\in\Lambda} a_l \sin(l\cdot\xi) \\ &= \sum_{k,l\in\Lambda} a_k a_l \cos\big((k-l)\cdot\xi\big),\end{aligned}$$

we immediately complete the proof of Lemma 2 by Lemma 1.

Lemma 3. Assume that $A(\xi_1, \cdots, \xi_n) = \sum\limits_{k\in\Lambda} a_k \exp(-ik\cdot\xi)$ $(\xi = (\xi_1, \cdots, \xi_n))$ satisfies

$$\sum_{\mu\in E^n} \left|A(\xi + \mu\pi)\right|^2 = 1, \tag{5}$$

where $a_k \in R$. Then $b_\mu^k = 0$ in (4) when all k_j $(j = 1, \cdots, n)$ are even but not all k_j are zero, where $k = (k_1, \cdots, k_n)$.

Proof. By (5) and (4),

$$\begin{aligned}1 = \sum_{\mu\in E^n} \left|A(\xi+\mu\pi)\right|^2 &= \sum_{k\in\Lambda} \sum_{\nu\in E_1^n} \sum_{\mu\in E^n} b_\nu^k \prod_{j=1}^{n} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot(\xi_j + \mu_j\pi)\right) \\ &= \sum_{k\in\Lambda} \sum_{\nu\in E_1^n} \sum_{\mu\in E^n} b_\nu^k \prod_{j=1}^{n} (-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right).\end{aligned} \tag{6}$$

Now we use induction to show the following formula:

$$\sum_{\mu\in E^n} \prod_{j=1}^{n} (-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) = \prod_{j=1}^{n} \left(1 + (-1)^{k_j}\right) \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right). \tag{7}$$

In fact, by a simple calculation, (7) holds when $n = 1$ and $n = 2$. Suppose that (7) holds when $n = m$. We now deal with the case of $n = m+1$. Observing that

$$\begin{aligned}&\sum_{\mu\in E^{m+1}} \prod_{j=1}^{m+1} (-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \\ =& \sum_{\mu=(\mu_1,\cdots,\mu_m,\mu_{m+1})\in E^{m+1}} \prod_{j=1}^{m} (-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \\ &\cdot (-1)^{k_{m+1}\mu_{m+1}} \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right) \\ =& \sum_{\mu=(\mu_1,\cdots,\mu_m,0)\in E^{m+1}} \prod_{j=1}^{m} (-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right)\end{aligned}$$

$$
\begin{aligned}
&\cdot (-1)^{k_{m+1}\mu_{m+1}} \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right) \\
&+ \sum_{\mu=(\mu_1,\cdots,\mu_m,1)\in E^{m+1}} \prod_{j=1}^{m}(-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \\
&\cdot (-1)^{k_{m+1}\mu_{m+1}} \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right) \\
=& \sum_{\mu=(\mu_1,\cdots,\mu_m)\in E^{m}} \prod_{j=1}^{m}(-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right) \\
&+ \sum_{\mu=(\mu_1,\cdots,\mu_m)\in E^{m}} \prod_{j=1}^{m}(-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \\
&\cdot (-1)^{k_{m+1}} \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right) \\
=& \sum_{\mu=(\mu_1,\cdots,\mu_m)\in E^{m}} \prod_{j=1}^{m}(-1)^{k_j\mu_j} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) \\
&\left[\left(1+(-1)^{k_{m+1}}\right) \cos\left(\nu_{m+1}\frac{\pi}{2} - k_{m+1}\cdot\xi_{m+1}\right)\right] \\
=& \prod_{j=1}^{m+1} \left(1+(-1)^{k_j}\right) \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right),
\end{aligned}
$$

thus (7) holds. It follows from (6) and (7) that

$$
\sum_{\nu=(\nu_1,\cdots,\nu_n)\in E_1^n} \sum_{k=(k_1,\cdots,k_n)\in\Lambda} b_\nu^k \prod_{j=1}^{n} \left(1+(-1)^{k_j}\right) \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right) = 1. \tag{8}
$$

By the linear independence of $\left\{ \prod_{j=1}^{n} \cos\left(\nu_j\frac{\pi}{2} - k_j\cdot\xi_j\right)\right\}$, we deduce that $b_\mu^k = 0$ if all k_j $(j = 1,\cdots,n)$ are even but not all k_j are zero. The proof is completed.

Lemma 4. Assume that $m_0(\xi_1,\cdots,\xi_n)$ has form (2) and satifies (1). Then

$$
\left|m_0(\xi_1,\cdots,\xi_n)\right|^2 = \sum_{k=(k_1,\cdots,k_n)\in\Lambda} b^k \prod_{j=1}^{n} \cos(k_j\xi_j).
$$

Proof. By Lemma 2,

$$
\left|m_0(\xi_1,\cdots,\xi_n)\right|^2 = \sum_{k\in\Lambda} \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n} \prod_{j=1}^{n} b_\mu^k \cos\left(\mu_j\frac{\pi}{2} - k_j\xi_j\right). \tag{9}
$$

Obviously, it is sufficient to verify that $b_\mu^k = 0$ when $\mu \neq (0,\cdots,0)$. Thus we assume that $\mu \neq (0,\cdots,0)$ in the following proof.

Because $m_0(\xi_1,\cdots,\xi_n)$ has the form (2), $\xi_p = \pi$ is the $2N_p$ multiple root of $\left|m_0(\xi_1,\cdots,\xi_n)\right|^2$, that is,

$$
\frac{\partial^l}{\partial\xi_p^l}|m_0(\xi_1,\cdots,\xi_n)|^2\,\Big|_{\xi_p=\pi} = 0, \qquad l = 0,\cdots,2N_p-1. \tag{10}
$$

By (9) and (10), $\forall\, l=0,\cdots,2N_p-1$,

$$\sum_{k\in\Lambda}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n}\ \prod_{j=1,j\neq p}^{n} b_\mu^k \cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)(-k_p)^l\cos\Big(\mu_p\frac{\pi}{2}-k_p\pi+l\frac{\pi}{2}\Big)=0. \quad (11)$$

So we deduce from (11) that

$$\sum_{k_p=0}^{2N_p-1}\ \sum_{k_j\in\Lambda_j,j\neq p}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n}\ \prod_{j=1,j\neq p}^{n} b_\mu^k \cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)(-k_p)^l\cos\Big(\mu_p\frac{\pi}{2}-k_p\pi+l\frac{\pi}{2}\Big)=0. \quad (12)$$

We sum the left-hand side of (12) separately over two summations:

$$\begin{aligned}&\sum_{k_j\in\Lambda_j,j\neq p}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n,\ \mu_p=1}\\ &\cdot\Big[\sum_{k_p=0}^{2N_p-1} b_\mu^k \prod_{j=1,j\neq p}^{n}\cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)(-k_p)^l\cos\Big(\frac{\pi}{2}-k_p\pi+l\frac{\pi}{2}\Big)\Big]\\ &+\sum_{k_j\in\Lambda_j,j\neq p}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n,\ \mu_p=0}\\ &\cdot\Big[\sum_{k_p=0}^{2N_p-1} b_\mu^k \prod_{j=1,j\neq p}^{n}\cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)(-k_p)^l\cos\Big(-k_p\pi+l\frac{\pi}{2}\Big)\Big]=0.\end{aligned}$$

It is not hard to know that the second term vanishes in the above formula when $l=1,3,\cdots,2N_p-1$. When $l=0,2,\cdots,2N_p-2$, the first term disappears. Therefore $\forall\, l=1,3,\cdots,2N_p-1$,

$$\sum_{k_j\in\Lambda_j,j\neq p}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n,\ \mu_p=1}\Big[\sum_{k_p=0}^{2N_p-1} b_\mu^k(-k_p)^l\sin\Big(\Big(\frac{l}{2}-k_p\Big)\pi\Big)\Big]\prod_{j=1,j\neq p}^{n}\cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)=0 \quad (13)$$

and $\forall\, l=0,2,\cdots,2N_p-2$,

$$\sum_{k_j\in\Lambda_j,j\neq p}\ \sum_{\mu=(\mu_1,\cdots,\mu_n)\in E_1^n,\ \mu_p=1}\Big[\sum_{k_p=0}^{2N_p-1} b_\mu^k(-k_p)^l\cos\Big(\Big(\frac{l}{2}-k_p\Big)\pi\Big)\Big]\prod_{j=1,\,j\neq p}^{n}\cos\Big(\mu_j\frac{\pi}{2}-k_j\xi_j\Big)=0. \quad (14)$$

So by the linear independence of $\{\prod_{j=1,j\neq p}^{n}\cos\big(\mu_j\frac{\pi}{2}-k_j\xi_j\big)\,\big|\,k_j\in\Lambda_j,\ j\neq p,\ \mu\in E_1^n\}$, we deduce that $\forall\, l=1,3,\cdots,2N_p-1$, $k_j\in\Lambda_j$, $j\neq p$, $\mu\in E_1^n$ and $\mu_p=1$,

$$\sum_{k_p=0}^{2N_p-1} b_\mu^k(-k_p)^l\sin\Big[\Big(\frac{l}{2}-k_p\Big)\pi\Big]=0 \quad (15)$$

and $\forall\, l = 0, 2, \cdots, 2N_p - 2$, $k_j \in \Lambda_j$, $j \neq p$, $\mu \in E_1^n$ and $\mu_p = 0$,

$$\sum_{k_p=0}^{2N_p-1} b_\mu^k (-k_p)^l \cos\left[\left(\frac{l}{2} - k_p\right)\pi\right] = 0. \tag{16}$$

For convenience, we introduce the following notations.

1. $\forall\, \mu = (\mu_1, \cdots, \mu_n) \in E^n$, define $\deg(\mu) = \sum_{j=1}^{n} \mu_j$.

2. $\forall\, k = (k_1, \cdots, k_n) \in \Lambda$, define $\Omega(k) = (\omega_1, \cdots, \omega_n)$, where $\omega_j = k_j - 2\left[\frac{k_j}{2}\right]$.

Obviously, $\Omega(k) \in E^n$, and all k_j $(j = 1, \cdots, n)$ are even in $k = (k_1, \cdots, k_n)\, (\in \Lambda)$ if and only if $\deg\big(\Omega(k)\big) = 0$. By Lemma 3, if $\deg\big(\Omega(k)\big) = 0$ and $k \neq (0, \cdots, 0)$, then $b_\mu^k = 0$.

$\forall\, r \geq 1$, assume that $b_\mu^k = 0$ when $\deg\big(\Omega(k)\big) = r - 1$. We will show that $b_\mu^k = 0$ when $\deg\big(\Omega(k)\big) = r$.

Set

$$S^p = \big\{(k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \,\big|\, k_j\ (j \neq p) \in \Lambda_j\big\}$$

and

$$S_r^p = \big\{k = (k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S^p \,\big|\, \deg\big(\Omega(k)\big) = r\big\}.$$

It is easy to show that $S^p = \bigcup_{r=0}^{n-1} S_r^p$, and that $\forall\, k = (k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S^p$, (15) holds when $\mu \in E^n$, $\mu \neq 0$ and $l = 1, 3, \cdots, 2N_p - 1$.

Denote

$$\Lambda^r = \big\{k = (k_1, \cdots, k_n) \in \Lambda \,\big|\, \deg\big(\Omega(k)\big) = r\big\}.$$

Then we immediately have

$$\Lambda^r \subset \bigcup_{p=1}^{n} \big\{k \in \Lambda \,|\, (k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S_{r-1}^p\big\}.$$

Thus, it is sufficient to prove that $b_\mu^k = 0$, $\forall\, k \in \big\{k \in \Lambda \,|\, (k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S_{r-1}^p\big\}$ $(p = 1, \cdots, n)$.

Obviously, $\forall\, k = (k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S_{r-1}^p$ $(p = 1, \cdots, n)$, then $k \in S^p$. Moreover, (15) also holds, that is, when $l = 1, 3, \cdots, 2N_p - 1$,

$$\sum_{k_p=0}^{2N_p-1} b_\nu^k (-k_p)^l \sin\left[\left(\frac{l}{2} - k_p\right)\pi\right] = 0.$$

Denote $l = 2t + 1$. By dividing the above formula into two summations, we have $\forall\, t = 0, \cdots, N_p - 1$,

$$\sum_{s=0}^{N_p-1} b_\mu^k (-1)^{2t+1} (2s+1)^{2t+1} \sin\left[\left(\frac{2t+1}{2} - (2s+1)\right)\pi\right] + \sum_{k_p \text{is even}} b_\mu^k (-k_p)^l \sin\left[\left(\frac{l}{2} - k_p\right)\pi\right] = 0. \tag{17}$$

Observe that when $(k_1, \cdots, k_{p-1}, k_{p+1}, \cdots, k_n) \in S_{r-1}^p$ and k_p is even,

$$\deg\ \big(\Omega\big((k_1, \cdots, k_{p-1}, k_p, k_{p+1}, \cdots, k_n)\big)\big) = r - 1,$$

hence $b_\mu^k = 0$ by the induction hypothesis. So, (17) can be rewritten in the following form:

$$\sum_{s=0}^{N_p-1} b_\mu^k (2s+1)^{2t+1} = 0, \qquad t = 0, \cdots, N_p - 1. \tag{18}$$

Since the matrix of coefficients is invertible in the above system of equations, $b_\mu^k = 0$. By induction, the proof of Lemma 4 is finished.

Lemma 5. If $m_0(\xi_1, \cdots, \xi_n)$ satisfies (1), then $|m_0(\xi_1, \cdots, \xi_n)|^2$ is separable for the variables $\xi_1, \cdots, \xi_n$.

Proof. Set

$$u_1 = \sin^2 \frac{\xi_1}{2}, \ \cdots, \ u_n = \sin^2 \frac{\xi_n}{2}.$$

Note that

$$\cos^n x = c_n^2 \cos^{n-2} x \sin^2 x + c_n^4 \cos^{n-4} x \sin^4 x - c_n^6 \cos^{n-6} x \sin^6 x + \cdots$$

and $\cos x = 1 - 2\sin^2 \frac{x}{2}$, hence we derive from Lemma 4 that

$$|m_0(\xi_1, \cdots, \xi_n)|^2 = P(u_1, \cdots, u_n),$$

where $P(u_1, \cdots, u_n)$ is a polynomial of variables $u_1, \cdots, u_n$. On the other hand, since $m_0(\xi_1, \cdots, \xi_n)$ has the form (2), we have

$$P(u_1, \cdots, u_n) = \prod_{j=1}^{n} (1 - u_j)^{N_j} Q(u_1, \cdots, u_n), \tag{19}$$

where $Q(u_1, \cdots, u_n)$ is also a polynomial of variables $u_1, \cdots, u_n$.

Define transforms: $T_0(x) = x$ and $T_1(x) = 1 - x$, where $x \in R$. Then $m_0(\xi, \cdots, \xi_n)$ satisfies (1) if and only if

$$\sum_{\mu=(\mu_1,\cdots,\mu_n)\in E^n} P(T_{\mu_1}(u_1), \cdots, T_{\mu_n}(u_n)) = 1. \tag{20}$$

In the following, we will use induction to show that equation (20) has a unique solution. When $n = 1$, (19) and (20) can be respectively reformulated as

$$P(u_1) = (1 - u_1)^{N_1} Q(u_1)$$

and

$$P(u_1) + P(1 - u_1) = 1.$$

Thus

$$(1 - u_1)^{N_1} Q(u_1) + u_1^{N_1} Q(1 - u_1) = 1.$$

By Theorem 6.1.1 in [1], the above equation has a unique solution. Suppose that the equation (20) has a unique solution when $n = m - 1$. Now we investigate the case of $n = m$. By (20),

$$\sum_{(\mu_1,\cdots,\mu_{m-1},0)\in E^m} P(T_{\mu_1}(u_1), \cdots, T_{\mu_{m-1}}(u_{m-1}), u_m)$$
$$+ \sum_{(\mu_1,\cdots,\mu_{m-1},1)\in E^m} P(T_{\mu_1}(u_1), \cdots, T_{\mu_{m-1}}(u_{m-1}), 1 - u_m) = 1.$$

It follows from (19) that

$$(1-u_m)^{N_m} \sum_{(\mu_1,\cdots,\mu_{m-1},0)\in E^m} \prod_{j=1}^{m-1} \big(1-T_{\mu_j}(u_j)\big)^{N_j} Q\big(T_{\mu_1}(u_1),\cdots,T_{\mu_{m-1}}(u_{m-1}),\, u_m\big)$$
$$+\, u_m^{N_m} \sum_{(\mu_1,\cdots,\mu_{m-1},1)\in E^m} \prod_{j=1}^{m-1} \big(1-T_{\mu_j}(u_j)\big)^{N_j} Q\big(T_{\mu_1}(u_1),\cdots,T_{\mu_{m-1}}(u_{m-1}),\, 1-u_m\big) = 1. \tag{21}$$

By [1, Theorem 6.1.1] and (21), we have

$$\sum_{(\mu_1,\cdots,\mu_{m-1})\in E^{m-1}} \prod_{j=1}^{m-1} \big(1-T_{\mu_j}(u_j)\big)^{N_j} Q\big(T_{\mu_1}(u_1),\cdots,T_{\mu_{m-1}}(u_{m-1}),\, u_m\big) = D_{N_m}(u_m), \tag{22}$$

where $D_{N_j}(u_j) = \sum_{l=0}^{N_j-1} d_l^j u_j^l$, $d_l^j = C_{N_j+l-1}^l$ with C_n^m indicating the combination number, that is, m elements being taken from n elements.

Set

$$Q(u_1,\cdots,u_m) = \sum_{l=0}^{N_m-1} Q_l(u_1,\cdots,u_{m-1}) u_m^l. \tag{23}$$

Then we deduce from (22) and (23) that $\forall\, l = 0,\cdots,N_m-1$,

$$\sum_{(\mu_1,\cdots,\mu_{m-1})\in E^{m-1}} \prod_{j=1}^{m-1} \big(1-T_{\mu_j}(u_j)\big)^{N_j} Q_l\big(T_{\mu_1}(u_1),\cdots,T_{\mu_{m-1}}(u_{m-1})\big) = d_l^m,$$

that is,

$$\sum_{(\mu_1,\cdots,\mu_{m-1})\in E^{m-1}} P_l^{m-1}\big(T_{\mu_1}(u_1),\cdots,T_{\mu_{m-1}}(u_{m-1})\big) = 1, \tag{24}$$

where

$$P_l^{m-1}(u_1,\cdots,u_{m-1}) = \frac{1}{d_l^m} \prod_{j=1}^{m-1} (1-u_j)^{N_j} Q_l(u_1,\cdots,u_{m-1})$$

and $l = 0,\cdots,N_m-1$.

By the induction hypothesis, the equation(24) has a unique solution P^{m-1}, that is,

$$P^{m-1} = P_l^{m-1}, \qquad l = 0,\cdots,N_m-1.$$

Therefore by (19) and (23),

$$P(u_1,\cdots,u_m) = (1-u_m)^{N_m} D_{N_m}(u_m) P^{m-1}(u_1,\cdots,u_m), \tag{25}$$

and (24) also holds when P_l^{m-1} in (24) is substituted by P^{m-1}. Thus by induction, the equation (20) has a unique solution.

Using induction again, we derive from (25) that

$$P(u_1,\cdots,u_n) = \prod_{j=1}^{n} (1-u_j)^{N_j} D_{N_j}(u_j).$$

Therefore $\big|m_0(\xi_1,\cdots,\xi_n)\big|^2$ is a separable variate. The proof of Lemma 5 is completed.

4. Proof of Theorem

Necessity. Suppose that $\phi(x_1,\cdots,x_n)$ is a tensor product of n Daubechies type scaling functions $\phi_1(x_1),\cdots,\phi_n(x_n)$. Then

$$\phi(x_1,\cdots,x_n)=\prod_{j=1}^{n}\phi_j(x_j).$$

Therefore it follows from the above formula and the definition of Daubechies type wavelet that

$$\widehat{\phi}(\xi_1,\cdots,\xi_n)=\prod_{j=1}^{n}\widehat{\phi}_j(\xi_j), \tag{26}$$

$$\widehat{\phi}_j(\xi_j)=m_j\Big(\frac{\xi_j}{2}\Big)\widehat{\phi}_j\Big(\frac{\xi_j}{2}\Big) \tag{27}$$

and

$$m_j(\xi_j)=\Big(\frac{1+\exp{(-i\xi_j)}}{2}\Big)^{N_j}Q_j\big(\exp(-i\,\xi_j)\big), \tag{28}$$

where $Q_j(x)=\sum\limits_{l=0}^{N_j-1}q_lx^l$ is a polynomial, $q_l\in R$ and $N_j\in Z^+$. From (26), (27) and (28), we deduce that

$$m_0(\xi_1,\cdots,\xi_n)=\prod_{j=1}^{n}m_j(\xi_j)=\prod_{j=1}^{n}\Big(\frac{1+e^{-i\xi_j}}{2}\Big)^{N_j}Q_j(e^{-i\xi_j})$$

$$=\prod_{j=0}^{n}\Big(\frac{1+e^{-i\xi_j}}{2}\Big)^{N_j}\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_n=0}^{N_n-1}a_{k_1\cdots k_n}\exp\Big(-i\sum_{l=1}^{n}k_l\xi_l\Big),$$

where $a_{k_1\cdots k_n}\in R$. This finishes the proof of necessity.

Sufficiency. Assume that the symbol function $m_0(\xi_1,\cdots,\xi_n)$ corresponding to $\phi(x_1\cdots,x_n)$ satisfies (2). Then we know from Lemma 5 that

$$\big|m_0(\xi_1,\cdots,\xi_n)\big|^2=\prod_{j=1}^{n}(1-u_j)^{N_j}D_{N_j}(u_j),$$

where $u_j=\sin^2\frac{\xi_j}{2}$. So

$$\big|m_0(\xi_1,\cdots,\xi_n)\big|^2\doteq\prod_{j=1}^{n}\Big(\cos^2\frac{\xi_j}{2}\Big)^{N_j}\big|Q_j\big(\exp{(-i\,\xi_j)}\big)\big|^2, \tag{29}$$

where

$$\big|Q_j(e^{-i\xi_j})\big|^2=D_{N_j}(u_j)\doteq\sum_{k_j=0}^{N_j-1}c_{N_j+k_j-1}^{k_j}\Big(\sin^2\frac{\xi_j}{2}\Big)^{k_j}.$$

By (29),

$$m_0(\xi_1,\cdots,\xi_n)=\pm\prod_{j=1}^{n}\Big(\frac{1+\exp(-i\xi_j)}{2}\Big)^{N_j}Q_j\big(\exp{(-i\xi_j)}\big)=\prod_{j=1}^{n}m_j(\xi_j). \tag{30}$$

So it follows from (30) that

$$\widehat{\phi}(\xi_1,\cdots,\xi_n)=\prod_{k=1}^{+\infty} m_0\Big(\frac{\xi_1}{2^k},\cdots,\frac{\xi_n}{2^k}\Big)$$
$$=\prod_{k=1}^{+\infty}\Big(\prod_{j=1}^{n} m_j\Big(\frac{\xi_j}{2^k}\Big)\Big)=\prod_{j=1}^{n}\Big(\prod_{k=1}^{+\infty} m_j\Big(\frac{\xi_j}{2^k}\Big)\Big)=\prod_{j=1}^{n}\widehat{\phi}_j(\xi_j).$$

Thus ϕ is obtained by the tensor product of ϕ_j $(j=1,2,\cdots,n)$, where $\phi_1,\cdots,\phi_{n-1}$ and ϕ_n are n Daubechies type scaling functions corresponding to the symbol functions $m_1,\cdots,m_{n-1}$ and m_n respectively. The proof of sufficiency is completed.

Acknowledgement. The authors would like to express their thanks to anonymous referees for many helpful suggestions to this paper, the first author thanks professor Zhan Yinwei for inviting him to visit Shantou University, during which time he performed part of the research for the results behind this paper.

References

1 I. Daubechies Ten Lectures on Wavelets. CBMS/NSF Series in Applied Mathematics, V.61, SIAM. Publ., 1992

2 A. Cohen, I. Daubechies. Nonseparable Bidimensional Wavelet Bases. *Rev. Math. Iberoamericana*, 1993, 9: 51–137

3 S.L. Peng. Characterization of Separable Bivariate Orthonormal Compactly Supported Wavelet Bases. *Acta Mathematics Sinica*, 2000, 43: 189–192

4 K. Gorchenig, W. Madych. Multiresolution Analysis, Haar Bases, and Self-similar Tilings. *IEEE Trans. Inform. Theory*, 1992, 38: 558–568

5 J. Kovacevic, M. Vetterli. Nonseparable Multidimensional Perfect Reconstruction Filterbanks. *IEEE Trans. Inform. Theory*, 1992, 38: 533–554

6 L. Villemoes. Continuity of Nonseparable Quincunx Wavelets. *Appl. Comput. Harmonic Anal.*, 1994, 1: 180–187

7 E. Belogay, Y. Wang. Arbitrarily Smooth Orthogonal Nonseparable Wavelets in R^2. *SIAM J. Math. Anal.*, 1999, 30(3): 678–697

8 W. He, M. Lai. Wavelet Apllications in Signal and Image Processing IV. Proceedings of SPIE, Vol. 3169: .1977, 303–314